

An Introduction to MUSIC and ESPRIT

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Based on

R. O. Schmidt, “Multiple emitter location and signal parameter estimation,”
IEEE Trans. Antennas & Propagation, vol. 34, no. 3, March 1986, and
R. Roy and T. Kailath, “ESPRIT – Estimation of signal parameters via rotation invariance
techniques,” *IEEE Trans. Acoust., Speech, Signal Proc.*, vol. 17, no. 7, July 1989

Introduction

- Well known high-resolution DOA algorithms
- Able to find DOAs of multiple sources
- High spatial resolution compared with other alg.
(I.e., a few antennas can result in high accuracy)
- MUSIC stands for **M**ultiple **S**ignal **C**lassifier
- ESPRIT stands for **E**stimation of **S**ignal
Parameters via **R**otational **I**nvariance **T**echnique
- Apply to only narrowband signal sources

Narrowband Signal Sources

- A complex sinusoid

$$s(t) = \alpha e^{j\beta} e^{j\omega t} = \rho e^{j\omega t}$$

- A real sinusoid is a sum of two sinusoids

$$\alpha \cos(\omega t + \beta) = \frac{\alpha}{2} e^{j\beta} e^{j\omega t} + \frac{\alpha}{2} e^{-j\beta} e^{-j\omega t} = \rho_1 e^{j\omega t} + \rho_2 e^{-j\omega t}$$

- A delay of a sinusoid is a phase shift

$$s(t - t_0) = e^{-j\omega t_0} \rho e^{j\omega t} = e^{-j\omega t_0} s(t)$$

- Apply approximately to narrowband signals

Narrowband Signal Sources

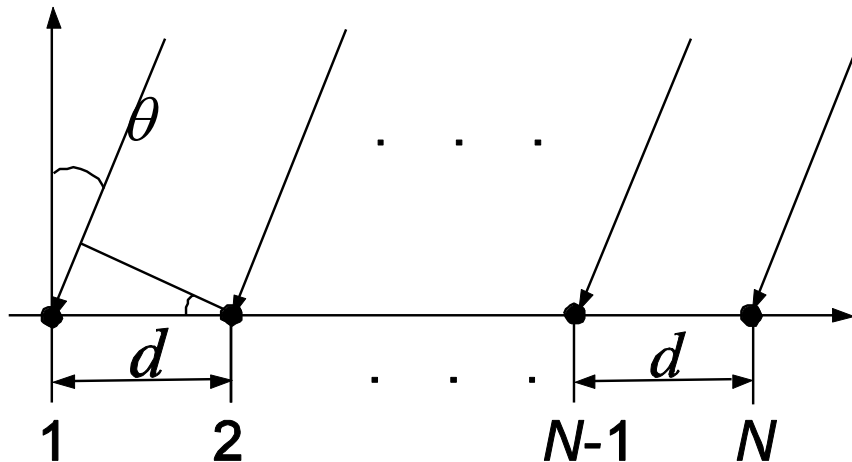
- Consider I narrowband signal sources

$$s_1(t) = \rho_1 e^{j\omega_1 t}, \quad s_2(t) = \rho_2 e^{j\omega_2 t}, \quad \dots, \quad s_I(t) = \rho_I e^{j\omega_I t}$$

- Assume that all frequencies are different
- Assume that all amplitudes are uncorrelated

$$E\{\rho_i \rho_j\} = \begin{cases} \sigma_i^2; & i = j \\ 0; & i \neq j \end{cases}$$

A Uniform Linear Array



A signal source $s(t) = \rho e^{j\omega t}$

“impinges” on the array

with an angle θ

c : propagation speed

- If the received signal at sensor 1 is $x_1(t) = s(t)$
- Then it is delayed at sensor i by $\Delta_i = \frac{(i-1)d \sin \theta}{c}$
- Then the received signal at sensor i is

$$x_i(t) = e^{-j\omega\Delta_i} s_1(t) = e^{-j\omega\Delta_i} s(t) = e^{-j\omega \frac{(i-1)d \sin \theta}{c}} s(t)$$

Signal Model

- Put received signals at all N sensors together:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-j\omega \frac{d \sin \theta}{c}} \\ e^{-j\omega \frac{2d \sin \theta}{c}} \\ \vdots \\ e^{-j\omega \frac{(N-1)d \sin \theta}{c}} \end{bmatrix} s(t) = \mathbf{a}(\theta) s(t)$$

- $\mathbf{a}(\theta)$ is called a “steering vector”

Signal Model

- If there are I sources signals received by the array, we get a “signal model”:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t)$$

$\mathbf{x}(t)$ --- received signal vector (N by 1)

$\mathbf{s}(t)$ --- source signal vector (I by 1)

$\mathbf{n}(t)$ --- noise vector (N by 1)

$\mathbf{A} = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_I)]$ (N by I)

$\mathbf{s}(t) = [s_1(t), \dots, s_I(t)]^T$

- Sources are independent, noises are uncorrelated
- Column of \mathbf{A} can also be normalized

The MUSIC Algorithm

- Compute the $N \times N$ correlation matrix

$$\mathbf{R}_x = E\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_0^2\mathbf{I}$$

where $\mathbf{R}_s = E\{\mathbf{s}(t)\mathbf{s}^H(t)\} = \text{diag}\{\sigma_1^2, \dots, \sigma_I^2\}$

- If the sources are somewhat correlated so \mathbf{R}_s is not diagonal, it will still work if \mathbf{R}_s has full rank.
- If the sources are correlated such that \mathbf{R}_s is rank deficient, then it is a problem. A common solution is “spatial smoothing”.

Q: Why is the rank of \mathbf{R}_s (being I) so important?

A: It defines the dimension of the signal subspace.

The MUSIC Algorithm

- For $N > I$, the matrix $\mathbf{AR}_s\mathbf{A}^H$ is singular, i.e.,

$$\det[\mathbf{AR}_s\mathbf{A}^H] = \det[\mathbf{R}_x - \sigma_0^2\mathbf{I}] = 0$$

- But this implies that σ_0^2 is an eigenvalue of \mathbf{R}_x
- Since the dimension of the null space of $\mathbf{AR}_s\mathbf{A}^H$ is $N-I$, there are $N-I$ such eigenvalues σ_0^2 of \mathbf{R}_x
- Since both \mathbf{R}_x and $\mathbf{AR}_s\mathbf{A}^H$ are non-negative definite, there are I other eigenvalues σ_i^2 such that $\sigma_i^2 > \sigma_0^2 > 0$
- Let \mathbf{u}_i be the i th eigenvector of \mathbf{R}_x corresponding to σ_i^2

$$\mathbf{R}_x\mathbf{u}_i = [\mathbf{AR}_s\mathbf{A}^H + \sigma_0^2\mathbf{I}]\mathbf{u}_i = \sigma_i^2\mathbf{u}_i; \quad i = 1, 2, \dots, N$$

$$\sigma_i^2 > \sigma_0^2 > 0, \quad i = 1, \dots, I; \quad \sigma_i^2 = \sigma_0^2, \quad i = I + 1, \dots, N$$

The MUSIC Algorithm

$$\mathbf{R}_x \mathbf{u}_i = [\mathbf{A} \mathbf{R}_s \mathbf{A}^H + \sigma_0^2 \mathbf{I}] \mathbf{u}_i = \sigma_i^2 \mathbf{u}_i; \quad i = 1, 2, \dots, N$$

- This implies

$$\mathbf{A} \mathbf{R}_s \mathbf{A}^H \mathbf{u}_i = (\sigma_i^2 - \sigma_0^2) \mathbf{u}_i; \quad i = 1, 2, \dots, N$$

$$\mathbf{A} \mathbf{R}_s \mathbf{A}^H \mathbf{u}_i = \begin{cases} (\sigma_i^2 - \sigma_0^2) \mathbf{u}_i; & i = 1, 2, \dots, I \\ 0; & i = I + 1, \dots, N \end{cases}$$

- Partition the N -dimensional vector space into the signal subspace \mathbf{U}_s and the noise subspace \mathbf{U}_n

$$[\mathbf{U}_s \quad \mathbf{U}_n] = \left[\underbrace{\mathbf{u}_1 \quad \dots \quad \mathbf{u}_I}_{\mathbf{U}_s: (\sigma_i^2 - \sigma_0^2) > 0 \text{ eigenvalues}} \quad \underbrace{\mathbf{u}_{I+1} \quad \dots \quad \mathbf{u}_N}_{\mathbf{U}_n: 0 \text{ eigenvalues}} \right]$$

The MUSIC Algorithm

- The steering vector $\mathbf{a}(\theta_i)$ is in the signal subspace
- Signal subspace is orthogonal to noise subspace

$$\mathbf{A}\mathbf{R}_s\mathbf{A}^H\mathbf{u}_i = \begin{cases} (\sigma_i^2 - \sigma_0^2)\mathbf{u}_i; & i = 1, 2, \dots, I \quad (1) \\ 0; & i = I + 1, \dots, N \quad (2) \end{cases}$$

- (1) means I linear combinations of columns of \mathbf{A} equal the signal subspace spanned by columns of \mathbf{U}_s
- (2) means the linear combinations of columns of \mathbf{A} , i.e., the signal subspace, is orthogonal to \mathbf{U}_n

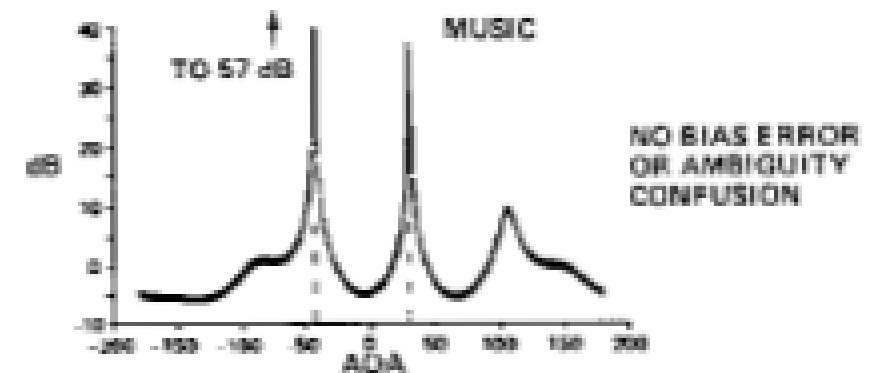
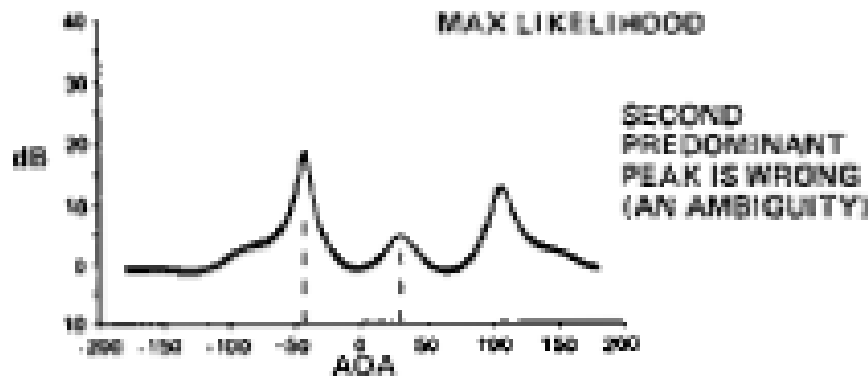
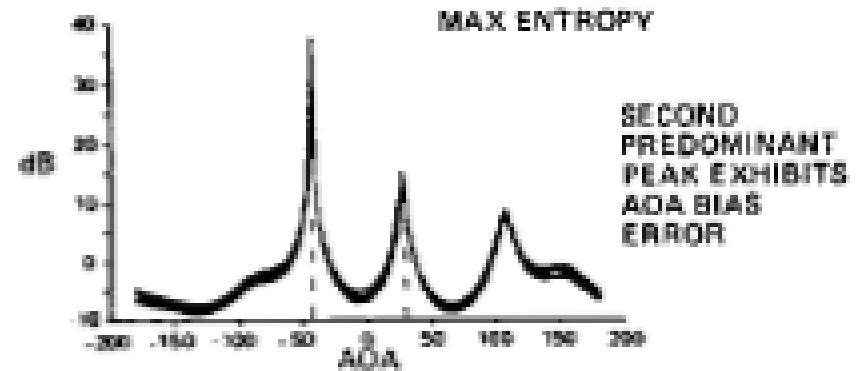
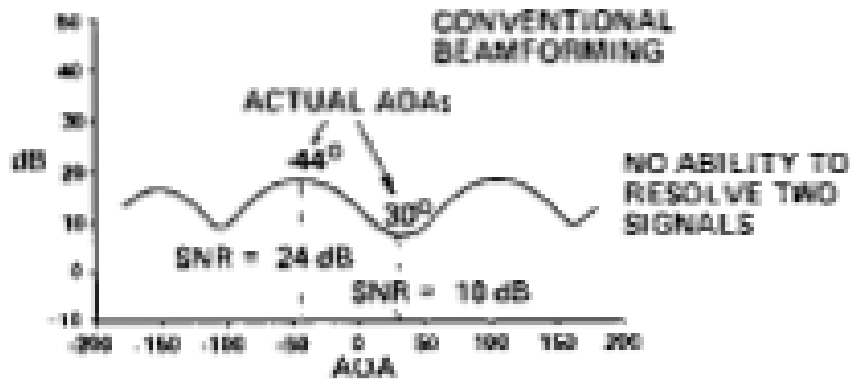
The MUSIC Algorithm

- The steering vector $\mathbf{a}(\theta_i)$ is in the signal subspace
- Signal subspace is orthogonal to noise subspace
- This implies that $\mathbf{a}^H(\theta_i)\mathbf{U}_n = \mathbf{0}$
- So the MUSIC algorithm searches through all angles θ , and plots the “spatial spectrum”

$$P(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{U}_n}$$

- Wherever $\theta = \theta_i$, $P(\theta)$ exhibits a peak
- Peak detection will give spatial angles of all incident sources

The MUSIC Algorithm



MUSIC spatial spectrum compared with other methods

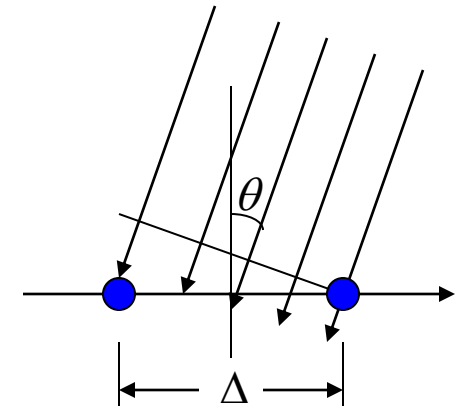
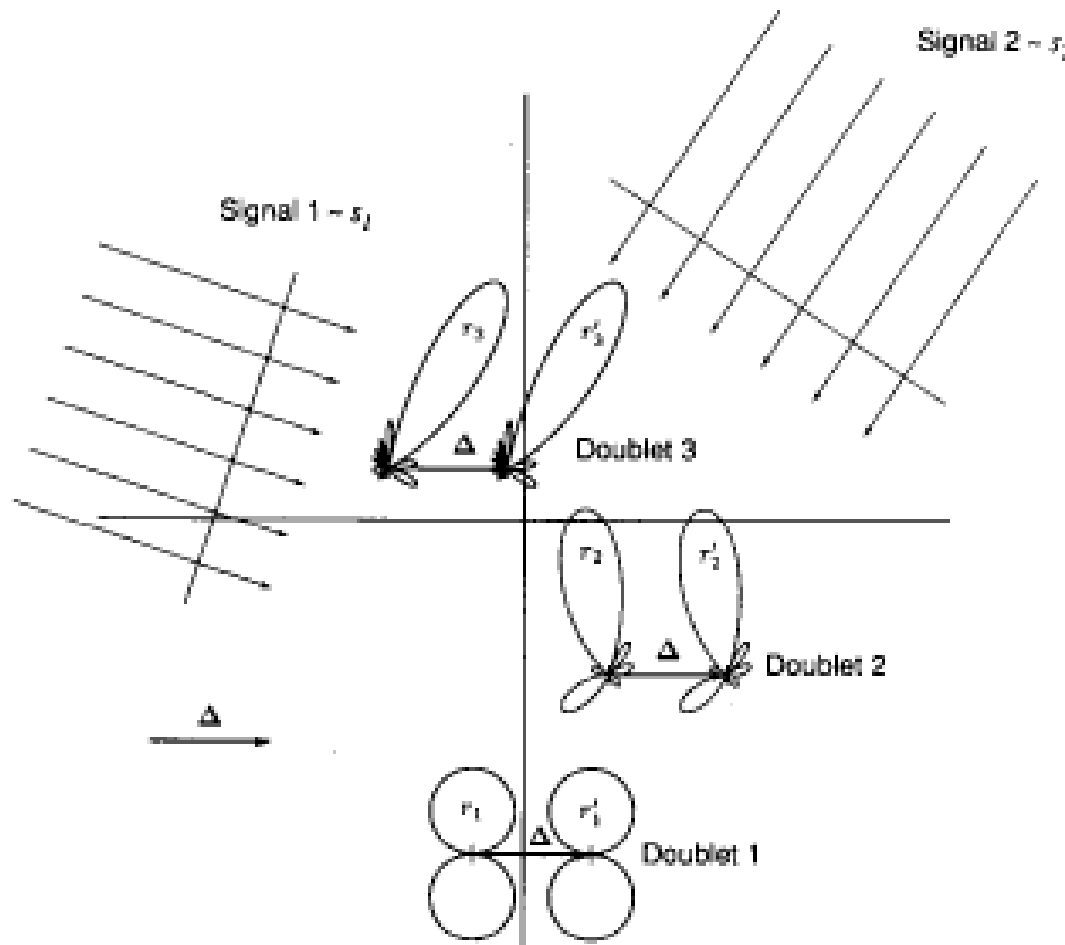
Pros/Cons of The MUSIC Algorithm

- Works for other array shapes, need to know sensor positions
- Very sensitive to sensor position, gain, and phase errors, need careful calibration to make it work well
- Searching through all θ could be computationally expensive
- The ESPRIT algorithm overcomes such shortcomings to some degree
- ESPRIT relaxes the calibration task somewhat
- ESPRIT takes much less computation
- But ESPRIT takes twice as many sensors

The ESPRIT Algorithm

- Based on “doublets” of sensors, i.e., in each pair of sensors the two should be identical, and all doubles should line up completely in the same direction with a displacement vector Δ having magnitude Δ
- Otherwise there are no restrictions, the sensor patterns could be very different from one pair to another
- The positions of the doublets are also arbitrary
- This makes calibration a little easier
- Assume N sets of doublets, i.e., $2N$ sensors
- Assume I sources, $N > I$

The ESPRIT Algorithm



The amount of delay between two sensors in each doublet for a given incident signal is the same for all doublets, which is

$$\Delta \sin \theta / c$$

A sensor array of doublets

The ESPRIT Algorithm

- This array is consisted of two identical subarrays, \mathbf{Z}_x and \mathbf{Z}_y , displaced from each other by Δ

$$\mathbf{x}(t) = \sum_{i=1}^I \mathbf{a}(\theta_i) s_i(t) + \mathbf{n}_x(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}_x(t)$$

$$\mathbf{y}(t) = \sum_{i=1}^I \mathbf{a}(\theta_i) e^{j\gamma_i} s_i(t) + \mathbf{n}_x(t) = \mathbf{A}\mathbf{\Phi}\mathbf{s}(t) + \mathbf{n}_x(t)$$

$$\gamma_i = \omega_0 \Delta \sin \theta_i / c \quad \mathbf{\Phi} = \text{diag.} \{ e^{j\gamma_1}, e^{j\gamma_2}, \dots, e^{j\gamma_I} \}$$

- The steering vector $\mathbf{a}(\theta)$ depends on the array geometry, and should be known just like in MUSIC

The ESPRIT Algorithm

- The objective is to estimate Φ , thereby obtaining θ_i

- Define

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\Phi \end{bmatrix} \mathbf{s}(t) + \begin{bmatrix} \mathbf{n}_x(t) \\ \mathbf{n}_y(t) \end{bmatrix} = \bar{\mathbf{A}}\mathbf{s}(t) + \mathbf{n}_z(t)$$

- Compute the $2N \times 2N$ correlation matrix

$$\mathbf{R}_z = E\{\mathbf{z}(t)\mathbf{z}^H(t)\} = \bar{\mathbf{A}}\mathbf{R}_s\bar{\mathbf{A}}^H + \sigma_0^2\mathbf{I}$$

- Since there are I sources, the I eigenvectors of \mathbf{R}_z corresponding to the I largest eigenvalues form the signal subspace \mathbf{U}_s ; The remaining $2N-I$ eigenvectors form the noise subspace \mathbf{U}_n

The ESPRIT Algorithm

- \mathbf{U}_s is $2N \times I$, and its span is the same as the span of $\bar{\mathbf{A}}$
- Therefore, there exists a unique nonsingular $I \times I$ matrix \mathbf{T} such that (\mathbf{A} needs to be known here)

$$\mathbf{U}_s = \bar{\mathbf{A}}\mathbf{T}$$

- Partition \mathbf{U}_s into two $N \times I$ submatrices

$$\mathbf{U}_s = \begin{bmatrix} \mathbf{U}_x \\ \mathbf{U}_y \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{T} \\ \mathbf{A}\Phi\mathbf{T} \end{bmatrix}$$

- The columns of both \mathbf{U}_x and \mathbf{U}_y are linear combinations of \mathbf{A} , so each of them has a column rank I

The ESPRIT Algorithm

- Define an $N \times 2I$ matrix, which has rank I

$$\mathbf{U}_{xy} = \begin{bmatrix} \mathbf{U}_x & \mathbf{U}_y \end{bmatrix}$$

- Therefore, \mathbf{U}_{xy} has a null space with dimension I , i.e., there exists a $2I \times I$ matrix \mathbf{F} such that

$$\mathbf{U}_{xy}\mathbf{F} = \mathbf{0} \Leftrightarrow \begin{bmatrix} \mathbf{U}_x & \mathbf{U}_y \end{bmatrix} \begin{bmatrix} \mathbf{F}_x \\ \mathbf{F}_y \end{bmatrix} = \mathbf{U}_x\mathbf{F}_x + \mathbf{U}_y\mathbf{F}_y = \mathbf{0}$$

$$\Leftrightarrow \mathbf{A}\mathbf{T}\mathbf{F}_x + \mathbf{A}\Phi\mathbf{T}\mathbf{F}_y = \mathbf{0} \Leftrightarrow \mathbf{A}\Phi\mathbf{T}\mathbf{F}_y = -\mathbf{A}\mathbf{T}\mathbf{F}_x$$

- Then the above gives, since \mathbf{T} has full column rank

$$\mathbf{A}\Phi\mathbf{T} = -\mathbf{A}\mathbf{T}\mathbf{F}_x\mathbf{F}_y^{-1} \Leftrightarrow \mathbf{A}\Phi = \mathbf{A}\mathbf{T}\mathbf{F}_x\mathbf{F}_y^{-1}\mathbf{T}^{-1} \Leftrightarrow \Phi = \mathbf{T}\mathbf{F}_x\mathbf{F}_y^{-1}\mathbf{T}^{-1}$$

The ESPRIT Algorithm

- The final algorithm is

$$\Phi = \mathbf{T}\mathbf{F}_x\mathbf{F}_y^{-1}\mathbf{T}^{-1}$$

- In practice, the measurement could be noisy, and there could be array calibration errors, so a total least squares ESPRIT is used
- The estimation is “one shot”, even with matrix inversions the computation is much less than the MUSIC search
- We must have $N > I$ for ESPRIT to work with $2N$ sensors, so need twice as many sensors as MUSIC

The ESPRIT Algorithm

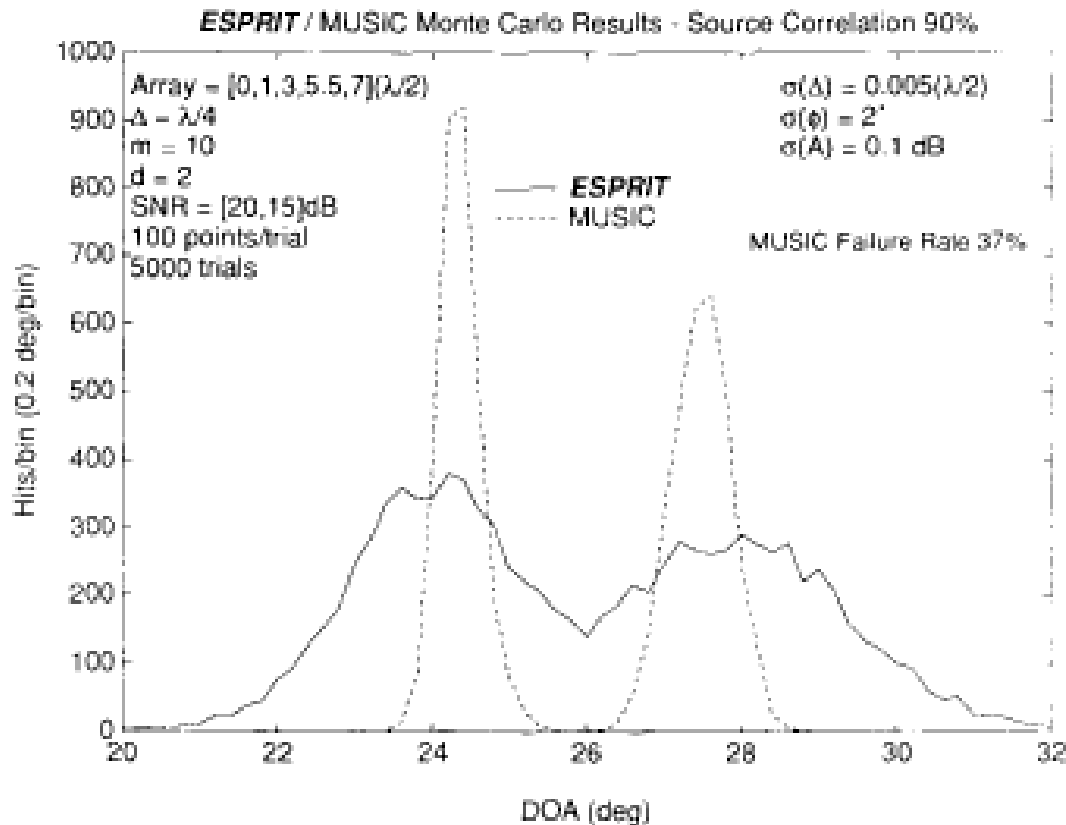


Fig. 4. Histogram of MUSIC and ESPRIT results—random 10-element linear array, source correlation 90 percent, small array aperture ($\Delta = \lambda/4$).

Simulation results of ESPRIT comparing with MUSIC

Conclusions

- Both methods are high-resolution, much better spatial resolution than beamforming and other methods
- Both methods are able to detect multiple sources
- If sensors are expensive and few, and if computation is not of concern, MUSIC is suitable
- If there are plenty of sensors compared with the number of sources to detect, and if computational power is limited, ESPRIT is suitable

About GIRD Systems, Inc.

- Founded in 2000 - based in Cincinnati
- Specializes in communications and signal processing, especially developing novel algorithms to solve challenging problems
- As of Nov. 2009, won more than 13 Phase I awards and 6 Phase II awards from Navy, Air Force, Army
- Partnerships with many large contractors including Northrop Grumman, L-3 Communications, etc.



About GIRD Systems, Inc.

- Key Technology Areas
 - Interference Mitigation (no reference, in-band)
 - Direction Finding (wideband, high-resolution)
 - Location/Navigation (Assisted GPS, GPS denied, signals of opportunity)
 - Wireless Network Security (physical layer)
 - Power Amplifier Linearization
 - Novel Communications systems/modeling
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